DISCRETE STRUCTURES
UNIT IV

Graph Theory

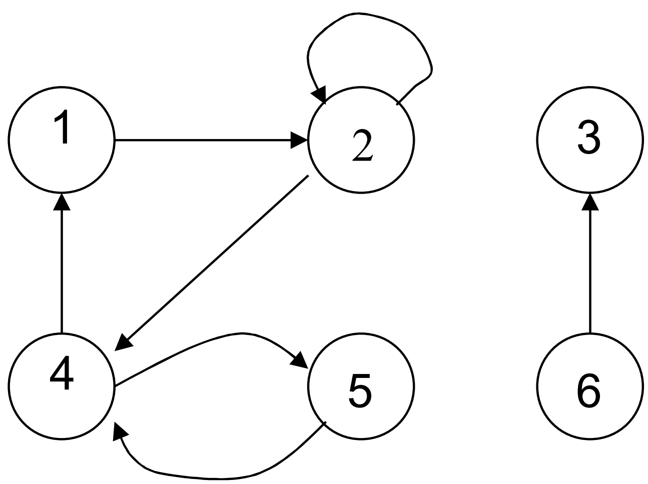
What is a Graph?

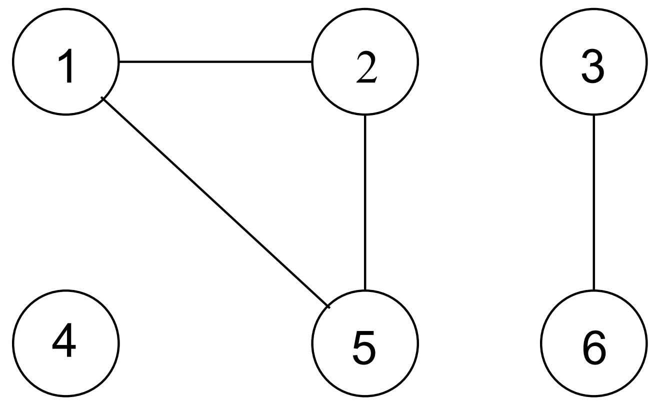
Informally a *graph* is a set of nodes joined by a set of lines or arrows.

A graph, written as G=G (V,E) consists of two components.

The finite set of vertices V , also called points or nodes

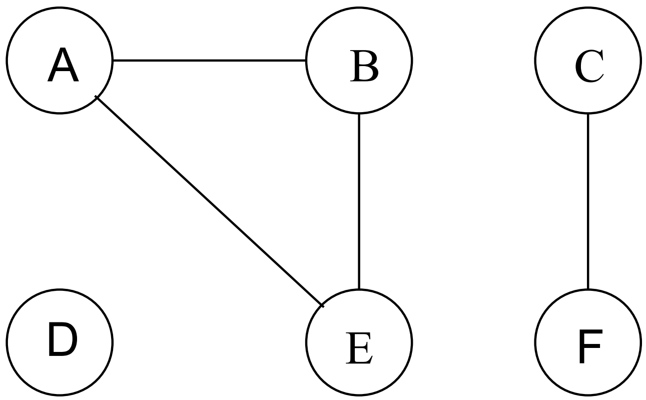
The finite set of (directed/undirected) edges E also called lines or arcs connecting pair of vertices.





Graph

An ***undirected graph*** $G = (V, E)$, but unlike a digraph the edge set E consist of unordered pairs. We use the notation (a, b) to refer to a directed edge, and $\{a, b\}$ for an undirected edge.



$V = \{ A, B, C, D, E, F \}$

$|V| = 6$

$E = \{ \{A, B\}, \{A,E\}, \{B,E\}, \{C,F\} \}$

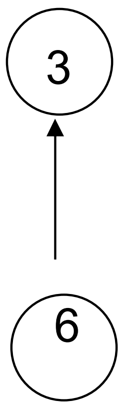
$|E| = 4$

Some texts use (a, b) also for undirected edges.
So (a, b) and (b, a) refers to the same edge.

DirectedGraph

A ***directed graph***, also called a ***digraph*** **G** is a pair (*V*, *E*), where the set *V* is a finite set and *E* is a binary relation on *V* . The set ***V*** is called the **vertex set** of *G* and the elements are called vertices. The set *E* is called the **edge set** of *G* and the elements are *edges* A

Edge from node *a* to node *b* is denote by the ordered pair (*a*, *b*).

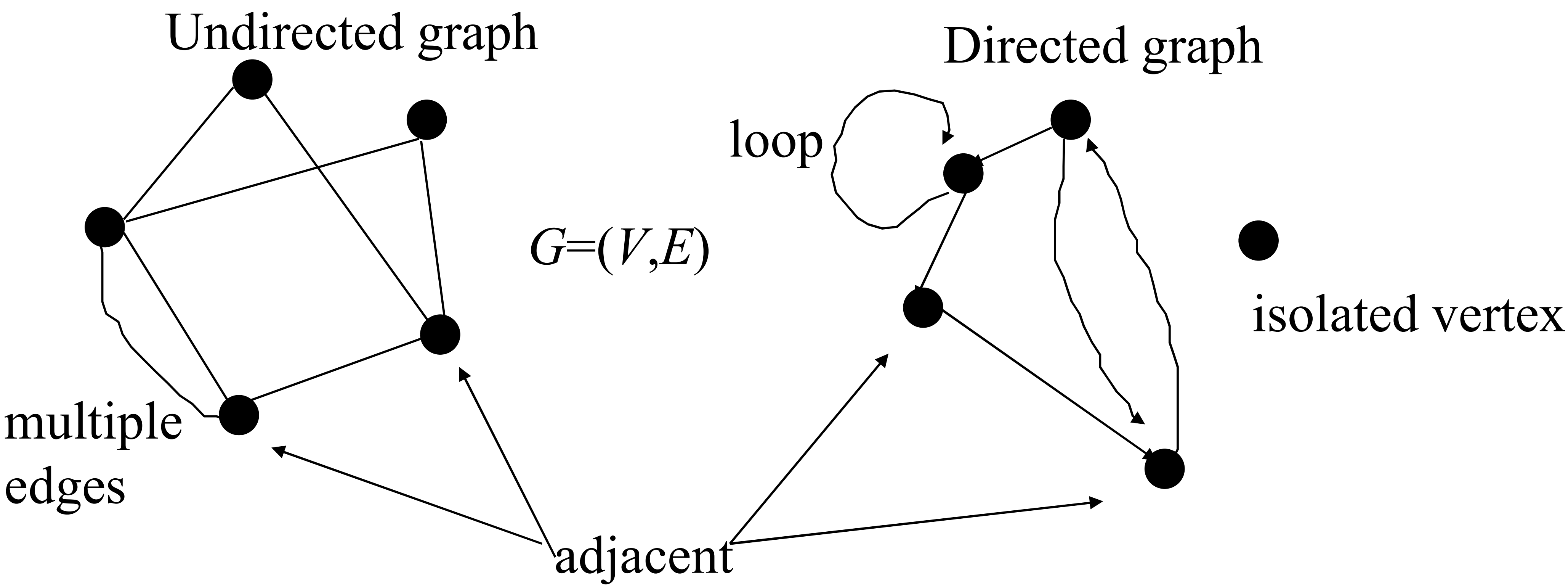


$V = \{ 1, 2, 3, 4, 5, 6, 7 \}$

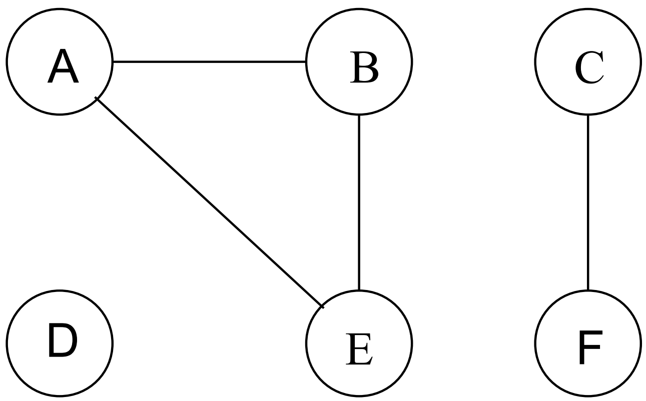
$| V | = 7$

$E = \{ (1,2), (2,2), (2,4), (4,5), (4,1), (5,4),(6,3) \}$

$| E | = 7$

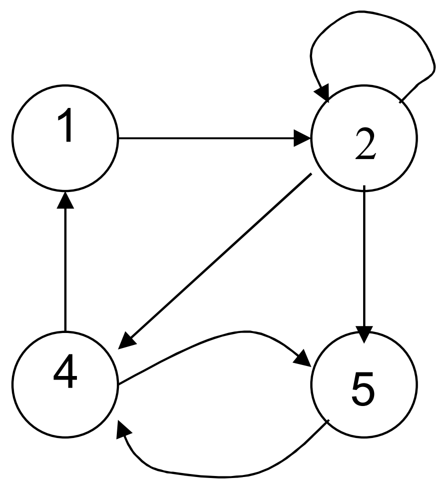

Undirected graph
Directed graph
loop
G=(V,E)
isolated vertex
multiple edges
adjacent

Degree of a Vertex in an undirected graph is the number of edges incident on it. In a directed graph , the ***out degree*** of a vertex is the number of edges leaving it and the ***in degree*** is the number of edges entering it.



The degree of B is 2.

Self-loop



The in degree of 2 is 2 and the out degree of 2 is 3.

Type Of Graphs

Simple Graph :- A simple graph G=(V,E) consists of V, a nonempty set of vertices and E, a set of unordered pair of distinct elements of V called edges.it has no self loop and parallel edges.

Multi Graph :-if in a directed or undirected graph there exists a certain pair of nodes that are joined by more then one edges such edges are called multiple edges or parallel edges and such graphs are called multigraph.

Peudograph :-general form of multigraph.in it loops are allowed.

Graph Summary

Type	Edges	Multiple Edges Allowed?	Loops Allowed?
Simple Graph	undirected	No	No
MultiGraph	undirected	yes	No
Pseudograph	undirected	yes	Yes
Directed Graph	directed	No	Yes
Directed Multigraph	directed	Yes	Yes

Graph Terminology

Adjacent vertex:- Two vertices are said to be adjacent if they are join by an edge.

If e={u,v} the edge e is called incident with the vertices u and v . The vertices u and v are called endpoints of the edge {u,v}

Loop:- An edge that is incident from and in to the same vertex is called loop.

Degree of Vertex:- The degree of vertex in a graph is number of edges connected with it. Degree is denoted by deg(v).

A vertex of degree zero is called isolated vertex

A vertex with degree one only is called pendant vertex.

in degree:- the number of edges ending at the vertex is called indegree of that particular vertex.

out degree:- the number of edges beginning from the vertex

Representation Of Graphs

We can use matrix for representing graphs in computer memory.

Incidence matrix : - The incidence matrix of a graph G with n vertices , e edges and no self loop is an n be e matrix M(G) = [mij], whose rows correspond to its vertices and columns corresponds to its edges. The element of the matrix will have values according to the following rule:

mij = 1 , if the j^{th} edge ej is incident on the i^{th} vertex vi

= 0 , otherwise.

Incidence matrix for digraph G is defined as:

mij = 1 , if the j^{th} edge ej is incident out of the i^{th} vertex vi

= 0 , if the j^{th} edge ej is not incident on the i^{th} vertex vi

= -1 , if the j^{th} edge ej is incident in to the i^{th} vertex vi

Conclusion from incidence matrix

As each edge is incident on two vertices , so each coloumn has two 1's (including -1)

The degree of vertex is equall to number of 1's in the corresponding row.

The row with all 0's is an isolated vertex.

Identical column represents present of parallel edges in the graph.

IMP POINT

1) Advantage of incident matrix is that it can represent parallel edges.

2) Disadvantage is that it can not represent self loops.

Adjacency Matrix

In it we make n by n vertices. A(G) = [aij] whose elements are defines as follows-

aij = 1, if there is an edge between ith and jth vertices.

= 0 , otherwise.

similarly , the adjacency matrix of a digraph G is defined as A(G) = [aij], such that

aij = 1 , if there is an edge directed from th vertex to jth vertex.

= 0 , otherwise.

Conclusion from Adjacency matrix

1) The entries along the principle diagonal of matrix A(G) are all 0's iff the graph has no loops.
2) It has no any provision for telling about parallel edges.
3) If there exists no loops and no parallel edges , then the degree of vertex is equal to the number of 1's in the corresponding row or column of A(G).

Walk

A walk is a sequence of vertices and edges that begins at Vi and travel along edges to Vj so that no edges appear more than once, however a vertex may appear more then once. in a walk ,first and last vertices in the sequence are called terminal vertex .

Closed And Open Walk:- A walk is said to be closed walk if it is possible that a walk begins and end at the same vertices. Otherwise the walk is called open walk. i.e. terminal vertices are different.

PATHS

A path is a walk through sequence ,V0,V1,V2….Vn of vertice ,each adjacent to the next ,without any repetition of vertices.if there exists a pth V0 to Vn in an undirected graph, then there always exists a path from Vn to V0 too. But in directed graph it is not necessary.

number of edges in a path is called length of the path.

TRAIL

A trail from a vertex u to v is path that does not invlove a repeated edge**.**

CIRCUITS

A circuit is a closed walk in which the terminal vertex coincides with the initial vertex and it contains no repeated edges.

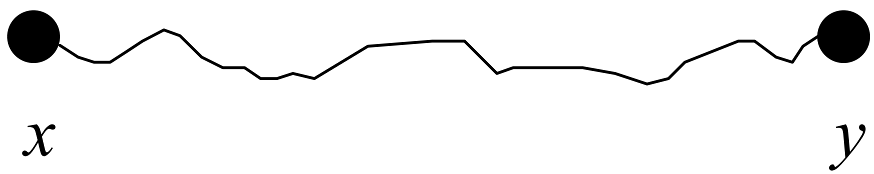
Simple circuit:- A circuit is said to be simple if it does not include the same edge twice.

Elementary Circuit:- if it does not meet the same vertex twice.

Fig 4.27 (page 197)

SUMMARY

TERM	REPEATED VERTEX	REPEATED EDGE	STARTS AND END AT THE SAME VERTEX
Trail	YES	NO	Yes
CIRCUIT	YES	NO	YES
PATH SIMPLE CLOSED	 NO YES	 NO YES	 NO YES



path: no vertex can be repeated

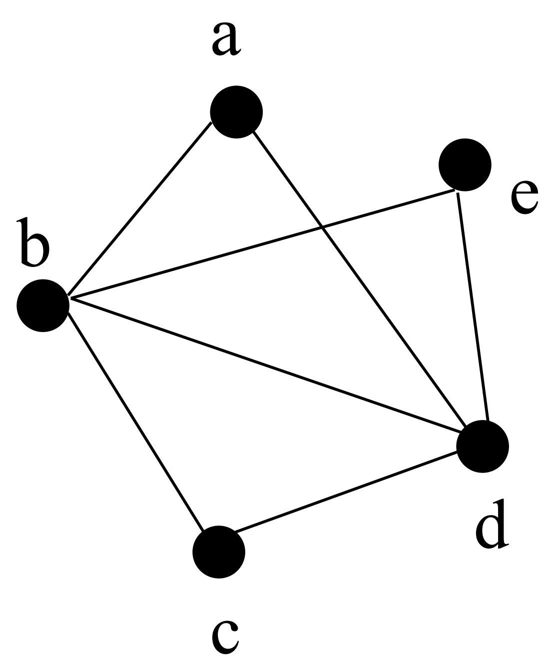
a-b-c-d-e

trail: no edge can be repeat

a-b-c-d-e-b-d

walk: no restriction

a-b-d-a-b-c



length: number of edges in this (path,trail,walk)

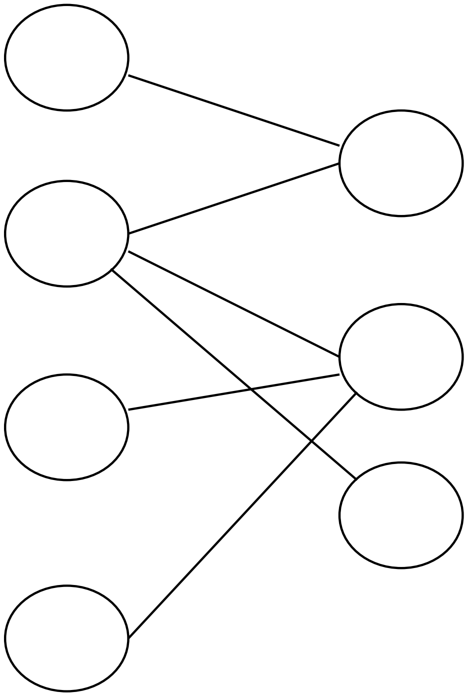
closed if *x*=*y*

closed trail: **circuit (a-b-c-d-b-e-d-a, one draw without lifting pen)**

closed path: **cycle (a-b-c-d-a)**

BIPARTITE GRAPH

A ***bipartite graph*** is an undirected graph $G = (V,E)$ in which V can be partitioned into 2 sets V_1 and V_2 such that $(u,v) \in E$ implies either $u \in V_1$ and $v \in V_2$ OR $v \in V_1$ and $u \in V_2$.



PLANAR GRAPH

A graph is said to be planar if it can be drawn on a plane in such a way that no edges cross one another, except of course at common vertices.

REGULAR GRAPH

A graph in which every vertex has the same degree is called a regular graph. If every vertex has degree r then graph is called a regular graph of degree r.

every null graph is regular of degree 0 and a complete graph Kn is regular of degree n-1.

if in a regular undirected graph, every vertex has same degree k then graph is called k-regular.

CONNECTED GRAPH

Connected undirected graph :-A graph is said to be connected if there exists atleast one path between every pair of its vertices, otherwise it is disconnected. Means for any given vertices u and v it is possible to travel from u to v along a sequence of adjacent edges of the graph.

ex j k sharma page 229

Distance :- in a connected graph G the distance between its vertices u and v is the length of the shortest path and is denoted by d (u, v).

Diameter:- in a connected Graph G the maximum distance between any two vertices is called diameter and is denoted by diam(G).

Cut point : - Vertex v in a connected graph G is called a cut point if G-v is disconnected . Where G-v is the graph obtained from G by deleting the vertex v and all edges connecting v.

Bridge:- An edge e of a connected graph G is called a bridge or cut edge if G-e is disconnected.

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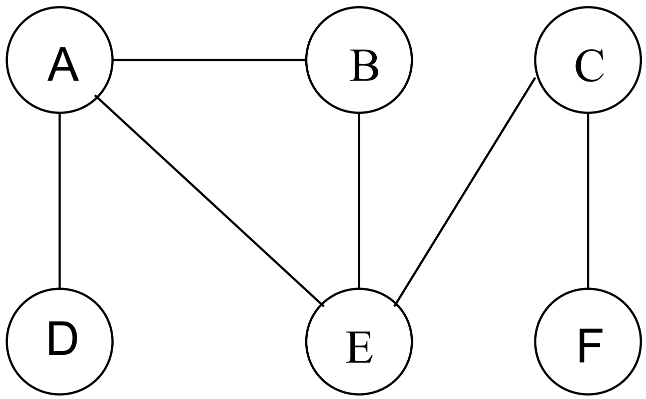
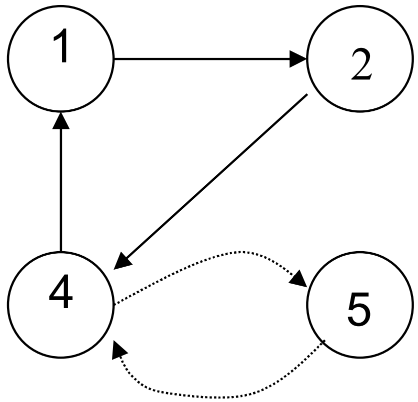
An undirected graph is ***connected*** if you can get from any node to any other by following a sequence of edges OR any two nodes are connected by a path.

A directed graph is ***strongly connected*** if there is a directed path from any node to any other node.





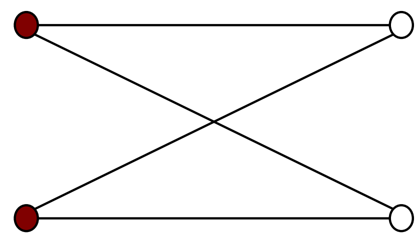
A graph is ***sparse*** if $| E | \approx | V |$

A graph is ***dense*** if $| E | \approx | V |^2$.

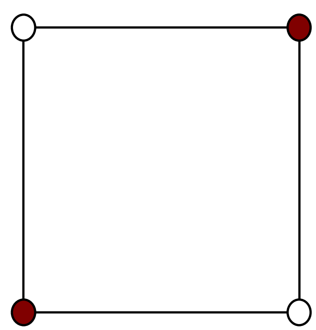
Graph Isomorphism

Intuitively, two graphs are isomorphic if can bend, stretch and reposition vertices of the first graph, until the second graph is formed. Etymologically, *isomorphic* means “same shape”.

EG: Can twist or relabel:



to obtain:



Graph Isomorphism Undirected Graphs

DEF: Suppose $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are pseudographs. Let $f : V_1 \rightarrow V_2$ be a function s.t.:

f is bijective

for all vertices *u,v* in V_1, the number of edges between *u* and *v* in G_1 is the same as the number of edges between $f(u)$ and $f(v)$ in G_2.

Then *f* is called an ***isomorphism*** and G_1 is said to be ***isomorphic*** to G_2.

Graph Isomorphism Digraphs

DEF: Suppose $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are directed multigraphs. Let $f : V_1 \rightarrow V_2$ be a function s.t.:

f is bijective

for all vertices u,v in V_1, the number of edges *from* u to v in G_1 is the same as the number of edges between $f(u)$ and $f(v)$ in G_2.

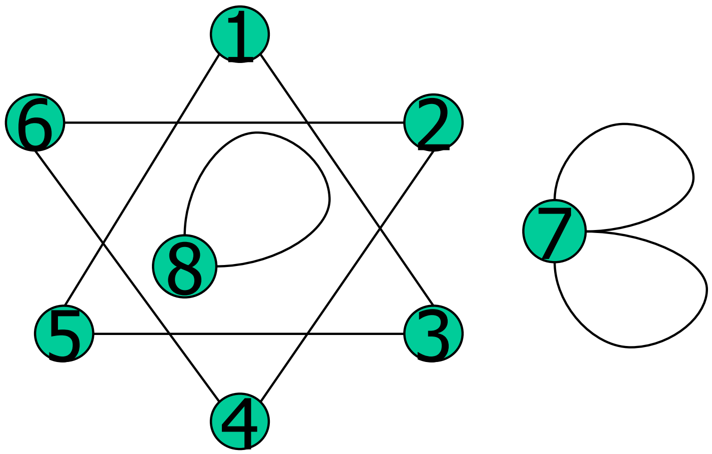
Then f is called an ***isomorphism*** and G_1 is said to be ***isomorphic*** to G_2.

Note: Only difference between two definitions is the italicized "from" in no. 2 (was "between").

Connected Components

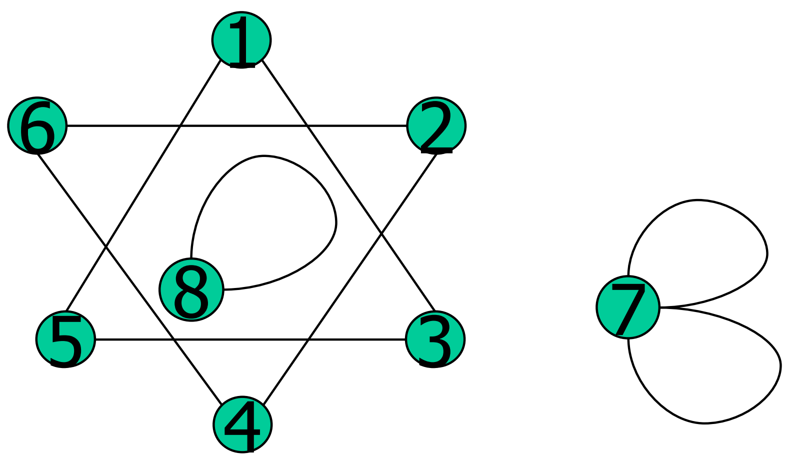
DEF: A ***connected component*** (or just ***component***) in a graph *G* is a set of vertices such that all vertices in the set are connected to each other and every possible connected vertex is included.

Q: What are the connected components of the following graph?



Connected Components

A: The components are {1,3,5},{2,4,6},{7} and {8} as one can see visually by pulling components apart:



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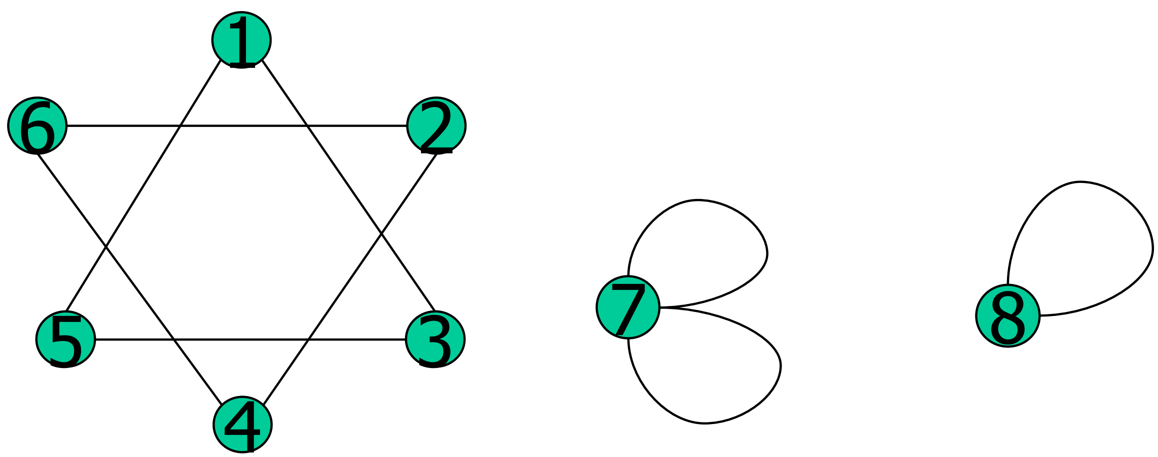

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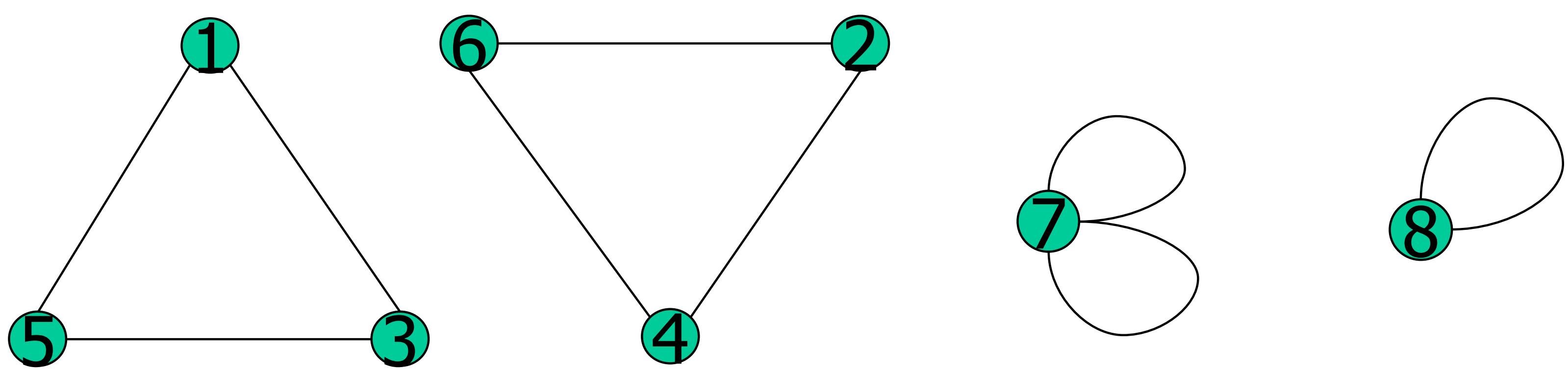


Connectivity in Directed Graphs

Resolution: Don't bother choosing which definition is better. Just define to separate concepts:

Weakly connected : can get from *a* to *b* in underlying undirected graph

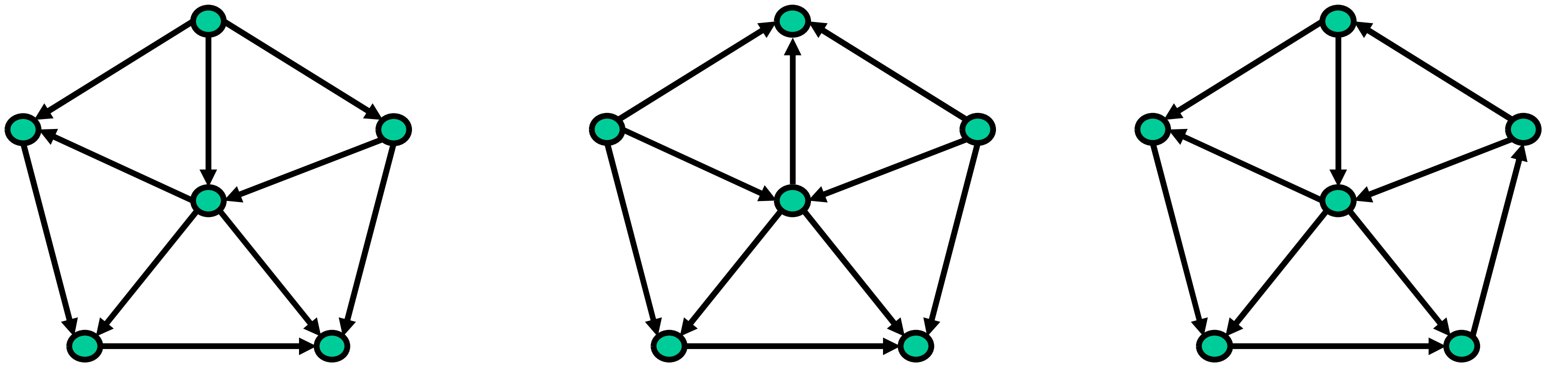
Semi-connected (my terminology): can get from *a* to *b* OR from *b* to *a* in digraph

Strongly connected : can get from *a* to *b* AND from *b* to *a* in the digraph

DEF: A graph is ***strongly*** (resp. ***semi***, resp. ***weakly***) connected if every pair of vertices is connected in the same sense.

Connectivity in Directed Graphs

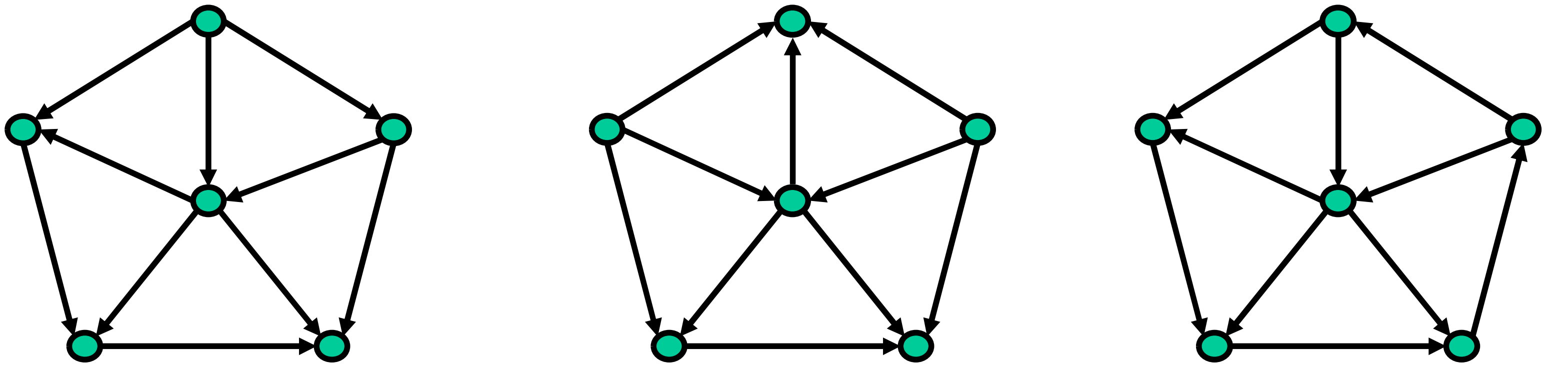
Q: Classify the connectivity of each graph.



Connectivity in Directed Graphs

A:

semi *weak* *strong*



EULERIAN GRAPH

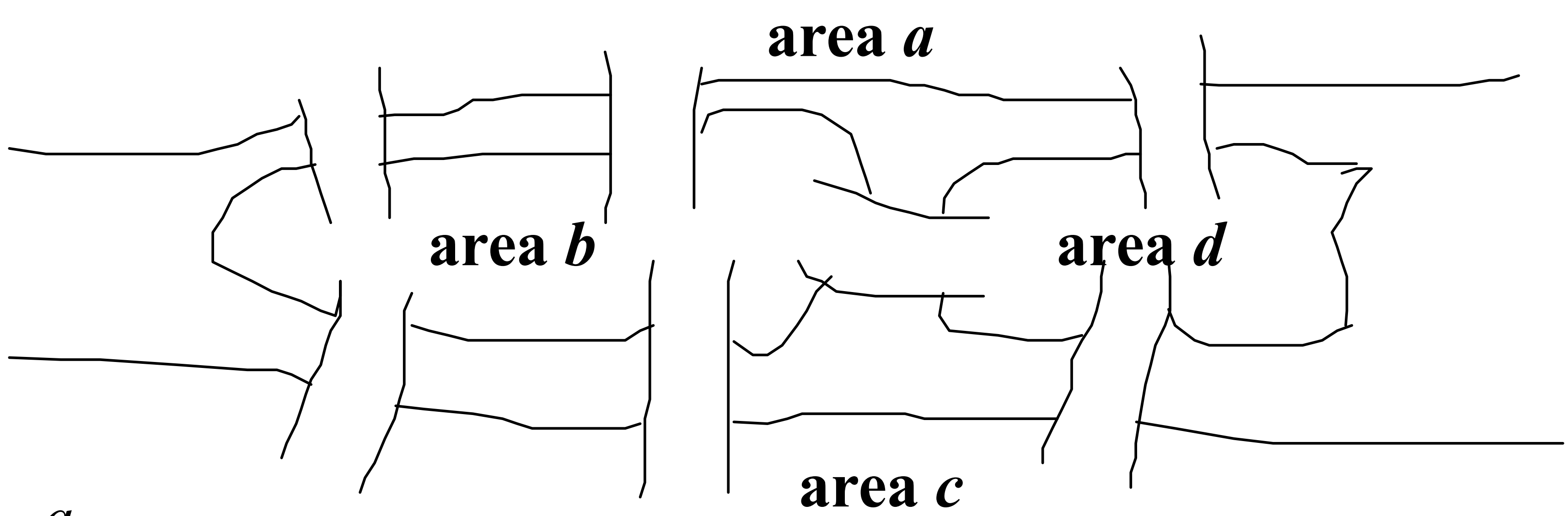
An undirected or multigraph G with no isolated vertices is said to have an eular circuit if there is a circuit in G that traverses every edge of the path exactly once. If there is an open trail from vertex u to v in G and this trail traverses every edge of the graph exactly once ,the trail is called eular trail

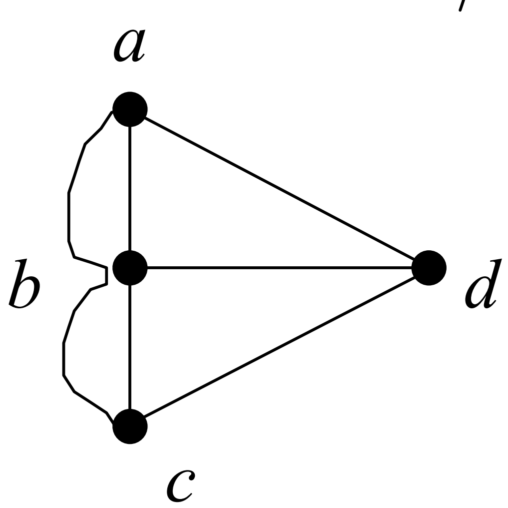
A path that passes through each edge exactly once but vertices may be repeated is called eular path. If the path is a circuit then it is called an eular circuit.

A graph that contains an euler tour (path or circuit) is called eulerian graph.

Vertex Degree: Euler Trails and Circuits

The Seven Bridge of Konigsberg





Find a way to walk about the city so as to cross each bridge exactly once and then return to the starting point.

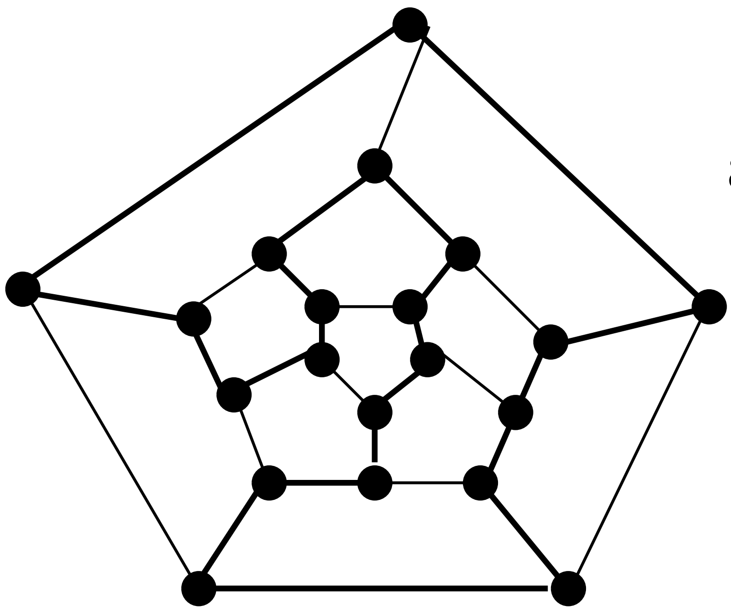
HAMILTONIAN GRAPH

Let G be a connected graph with |V| > 3. if there is a path in G that uses each vertex of the graph exactly once , then such a path is called Hamiltonian path.

If the path is a circuit that contain each vertex in G exactly once , except initial vertex that appears twice as the terminal vertex , then such path is called a Hamiltonian circuit or cycle.

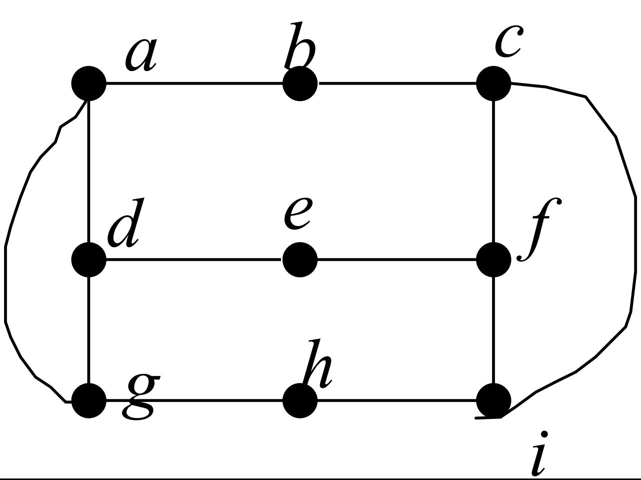
A graph with a closed path that includes every vertex exactly once is called a Hamiltonian graph

Hamilton Paths and Circuit



a path or cycle that contain every vertex

Unlike Euler circuit, there is **no known necessary and sufficient condition** for a graph to be Hamiltonian.



There is a Hamilton path, but no Hamilton cycle.

GRAPH COLORING

Problem:- Suppose a Graph G with n nodes is given . It is required to paint its nodes such that no two adjacent nodes will be of the same color. What is the minimum numbers of color required? This is called coloring problem.

Assigning all the nodes of a graph with colors such that no two adjacent nodes are assigned the same color is called proper or simple coloring. A graph in which every vertex has been assigned a color according to a proper coloring is called properly colored graph.

generally we use minimum number of colors for proper coloring of a graph.

A Graph G that requires minimum k different colors for its proper coloring is known as k-chromatic or k-colorable and number k is called chromatic number of G. symbolically chromatic number of a graph G is written as

k (G).

1) A complete graph is a graph where each vertex is connected to every other vertex.

for complete graph Kn chromatic number is n.

2) A cycle graph is a graph that consists of a single cycle or we can say that all the vertices connected in a single chain.The cycle graph with n vertices is denoted as Cn. The number of vertices in Cn equals the number of edges , and every vertex has degree 2 .

For cyclic Graph Cn (n>1) **3 if n is odd**

2 if n is even

3) The n star graph is a graph consisting of n nodes with one node having degree n-1 , and the other n-1 nodes having degree 1 .

for star graph Cn (n>1) chromatic number is 2.

4) A wheel graph Wn is a graph with n vertices , formed by connecting a single vertex to all vertices of a cycle having n-1 vertices

For Wheel Graph Wn, n>2   3 if n is odd

4 if n is even

Some observations are:-

1) A graph which consists of only isolated vertices has chromatic number 1.
2) A graph with one or more edges has a minimum chromatic number 2
3) A graph consists of simply one circuit with n>=3 vertices has chromatic number 2 if n is even and 3 if n is odd.
4) A complete bipartite graph Km,n has chromatic number 2
5) If d is the maximum degree of the vertices in a Graph G ,then

Chromatic number of G $\leq$ d+1.

APPLICATION OF GRAPH COLORING

An examination controller of a university needs to schedule the graduate examination. This problem can be solve by Graph coloring problem

Ans:- construct a graph where the vertices represent the courses . Connect an edge between two vertices if there is a common student they represent . Each time slot for an examination is represented by a different color. A scheduling of the examination will correspond to a coloring of the associated graph.

TREE

DEFINITION

A tree is a collection of nodes

- The collection can be empty
- (recursive definition) If not empty, a tree consists of a distinguished node r (the *root*), and zero or more nonempty *subtrees* T_1, T_2, ..., T_k, each of whose roots are connected by a directed *edge* from r

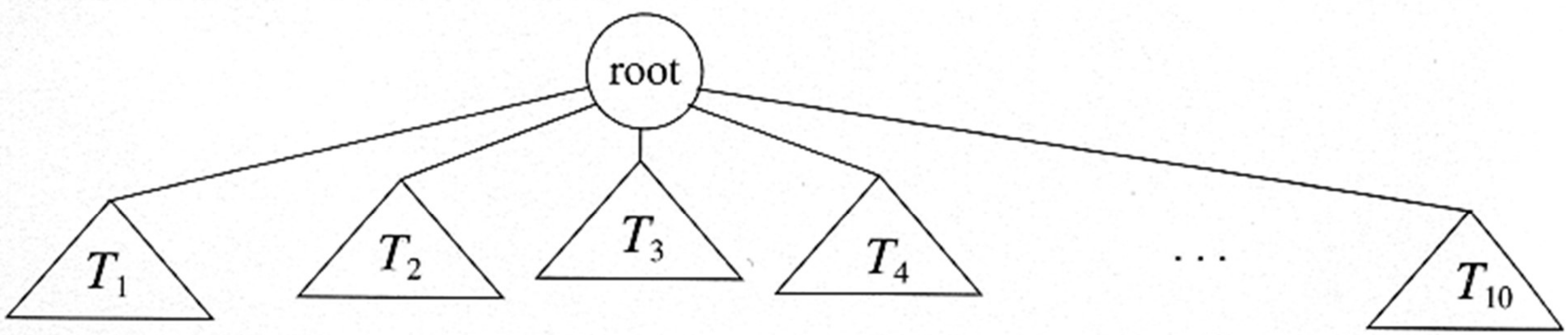


Figure 4.1 Generic tree

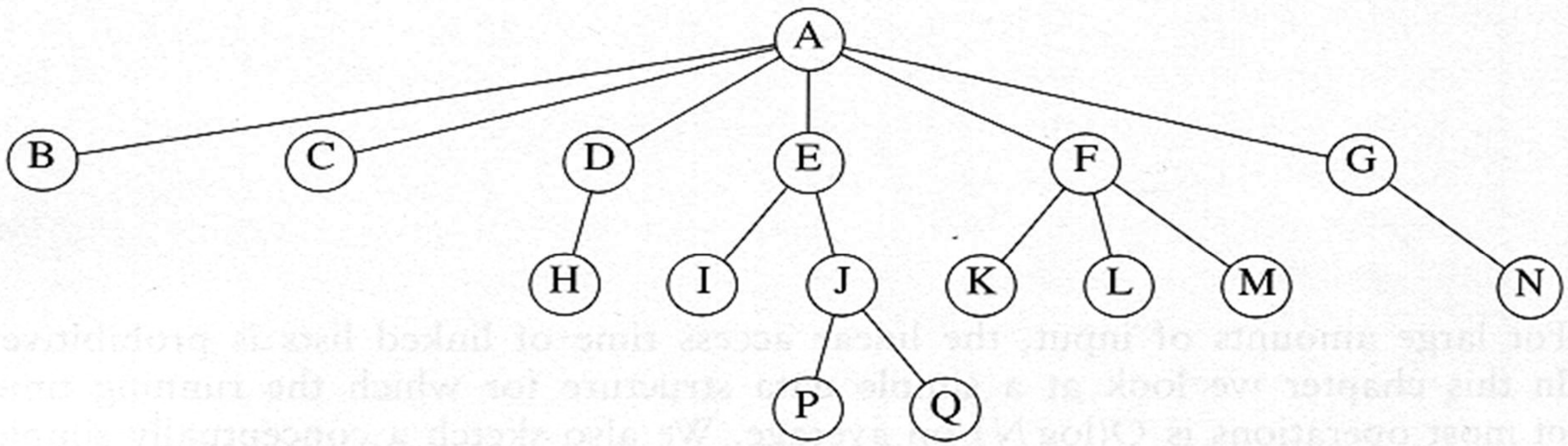


Figure 4.2 A tree

Child* and *parent

Every node except the root has one parent

A node can have an arbitrary number of children

Leaves

Nodes with no children

Sibling

nodes with same parent

Length

- number of edges on the path

Depth of a node

- length of the unique path from the root to that node
- The depth of a tree is equal to the depth of the deepest leaf

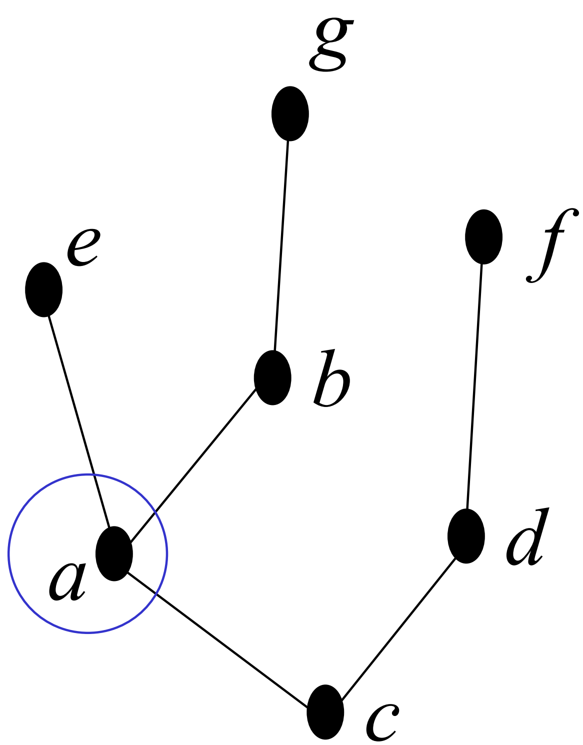
Height of a node

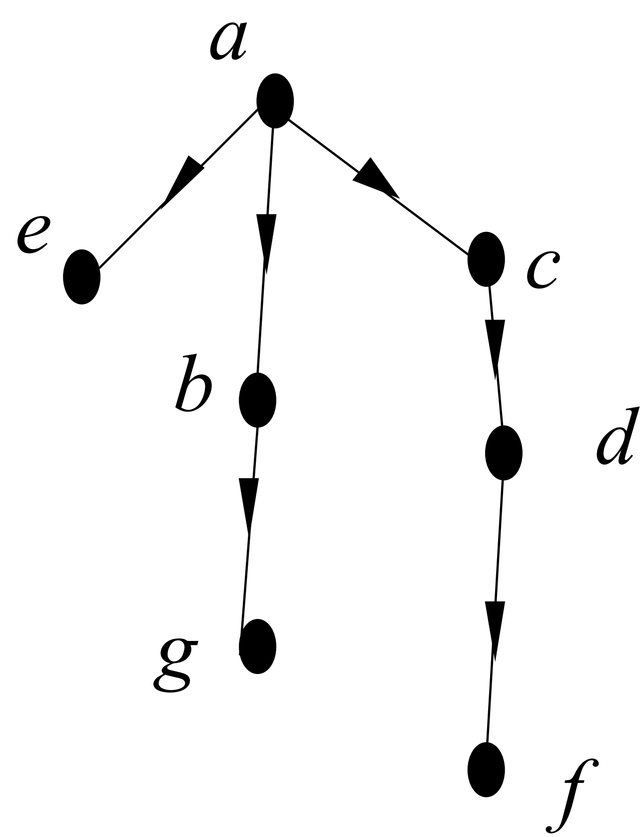
- length of the longest path from that node to a leaf
- all leaves are at height 0
- The height of a tree is equal to the height of the root

Ancestor and *descendant*

- *Proper ancestor* and *proper descendant*

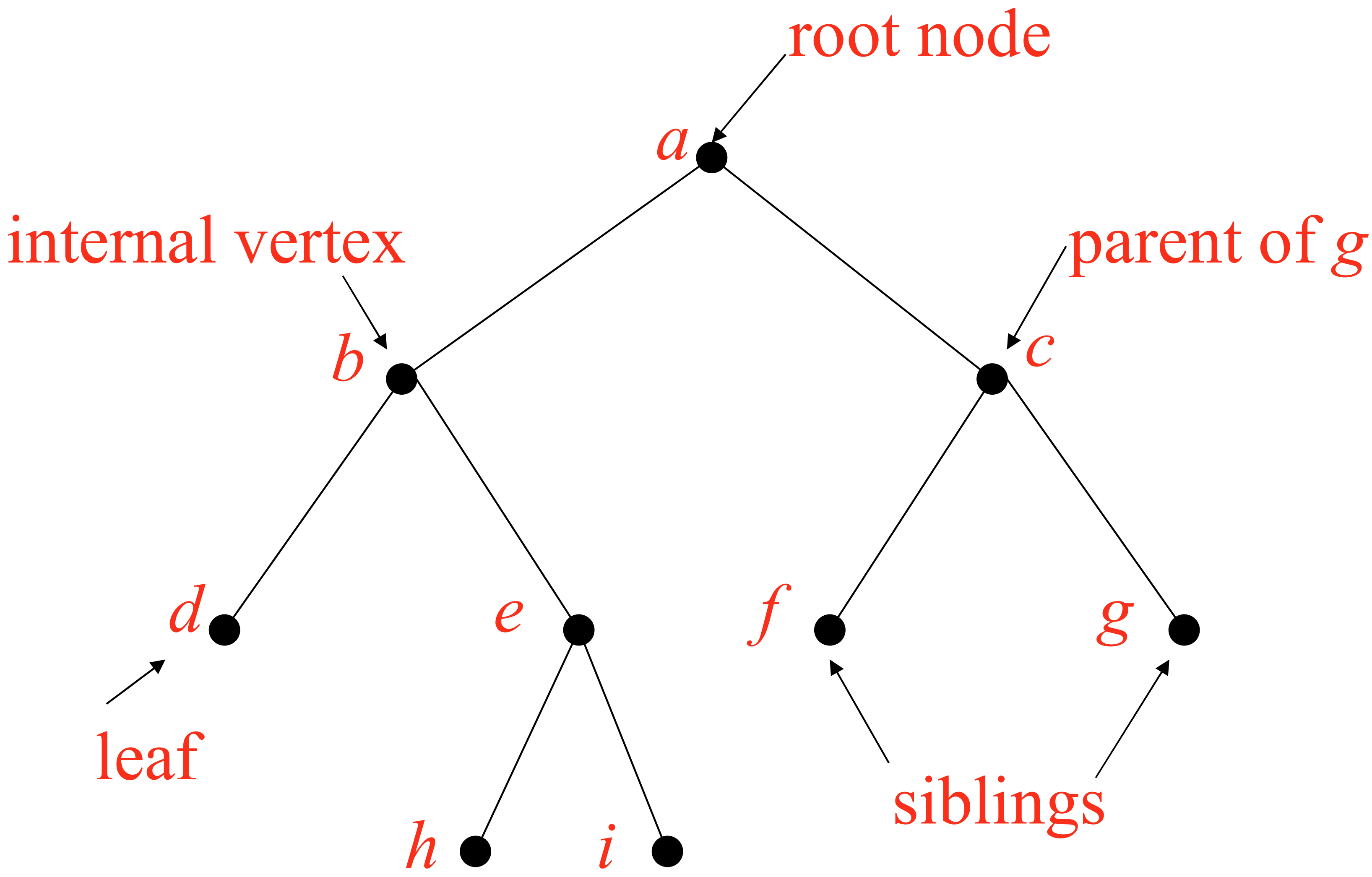
Rooted Trees

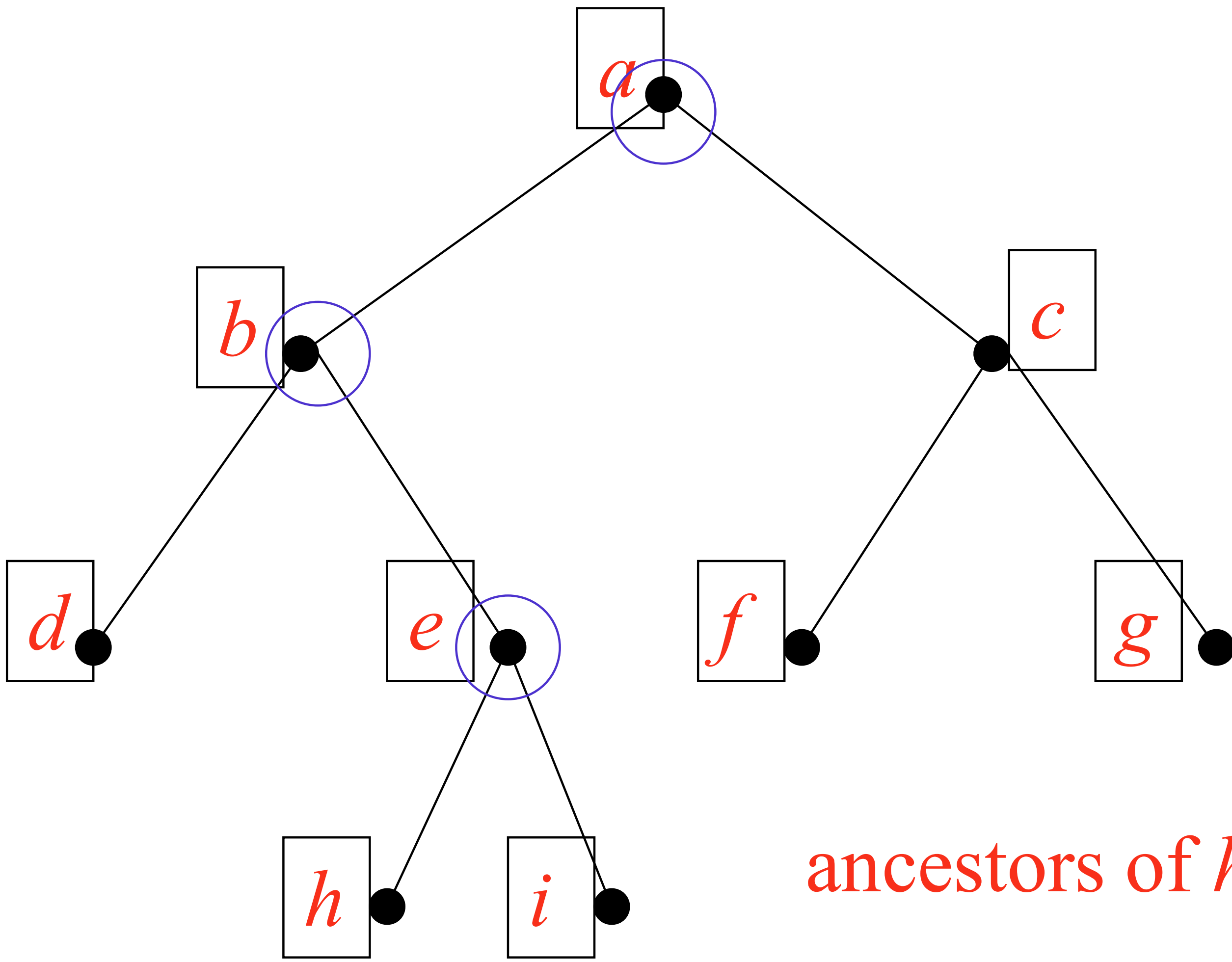




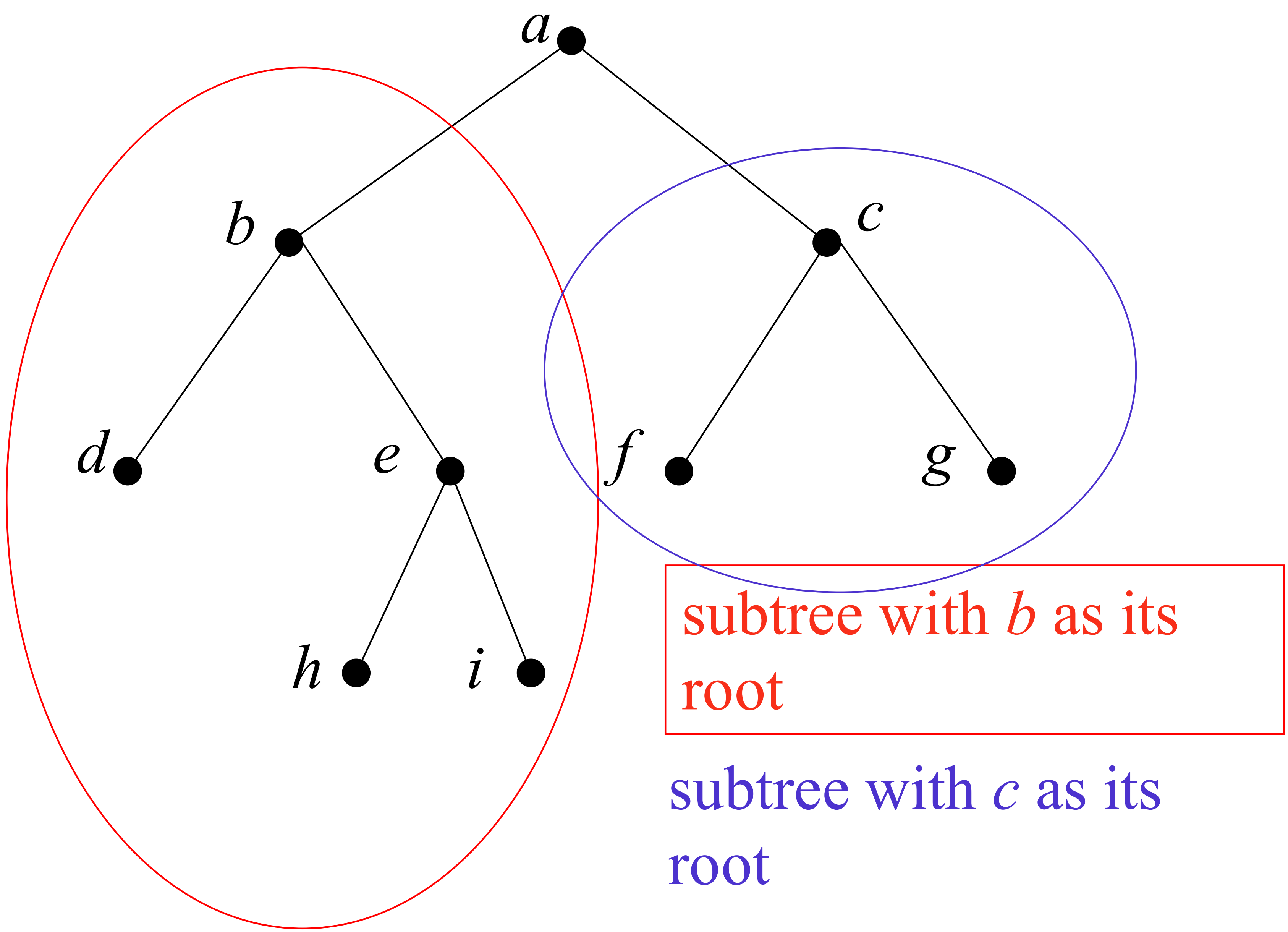
Rooted Trees

Once a vertex of a tree has been designated as the *root* of the tree, it is possible to assign direction to each of the edges.


root node
a
internal vertex
parent of g
b
c
d
e
f
g
leaf
siblings
h
i

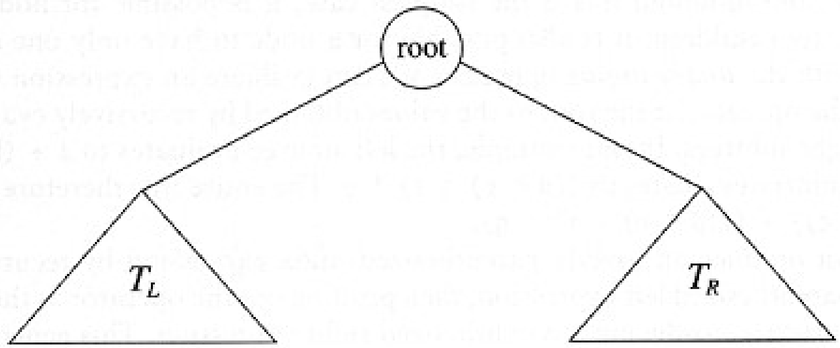


ancestors of *h* and *i*


a
b
c
d
e
f
g
h
i
subtree with *b* as its root
subtree with *c* as its root

Binary Trees

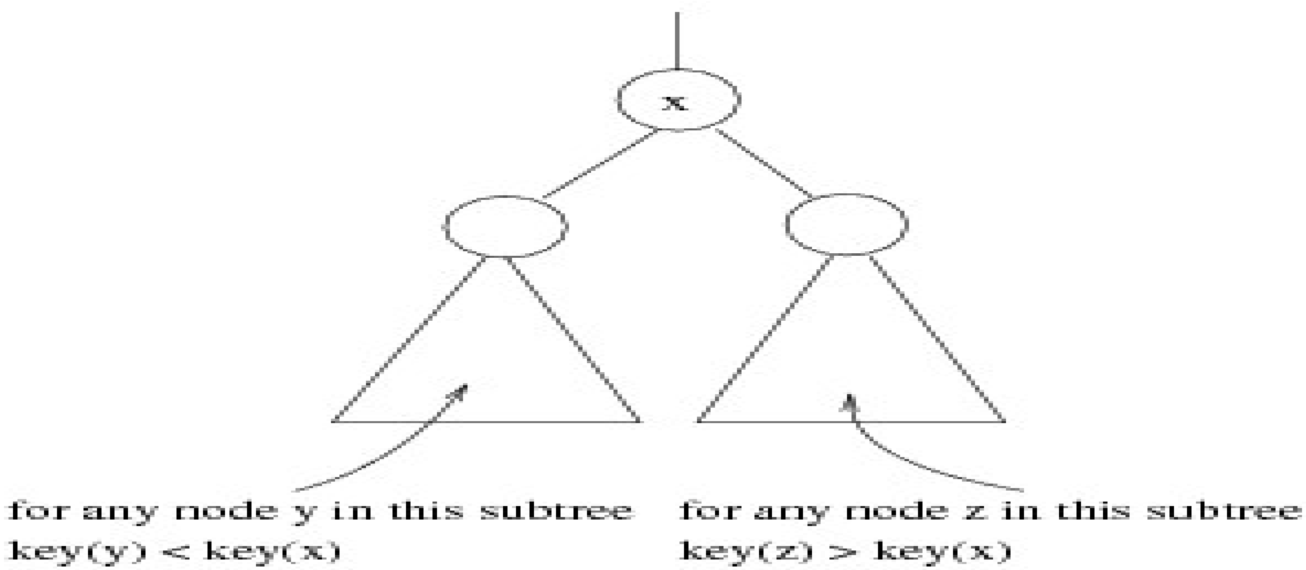
A tree in which no node can have more than two children



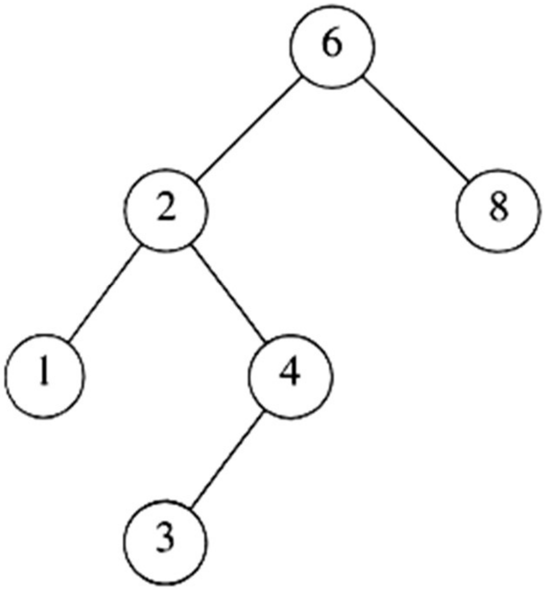
Binary Search Trees

Binary search tree property

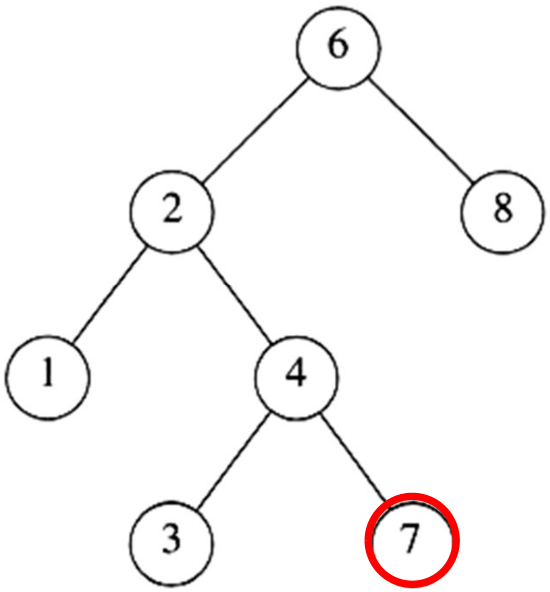
- For every node X, all the keys in its left subtree are smaller than the key value in X, and all the keys in its right subtree are larger than the key value in X



Binary Search Trees



A binary search tree



Not a binary search tree

Preorder, Postorder and Inorder

Preorder traversal

- node, left, right
- prefix expression
  - ✓ ++a*bc*+*defg

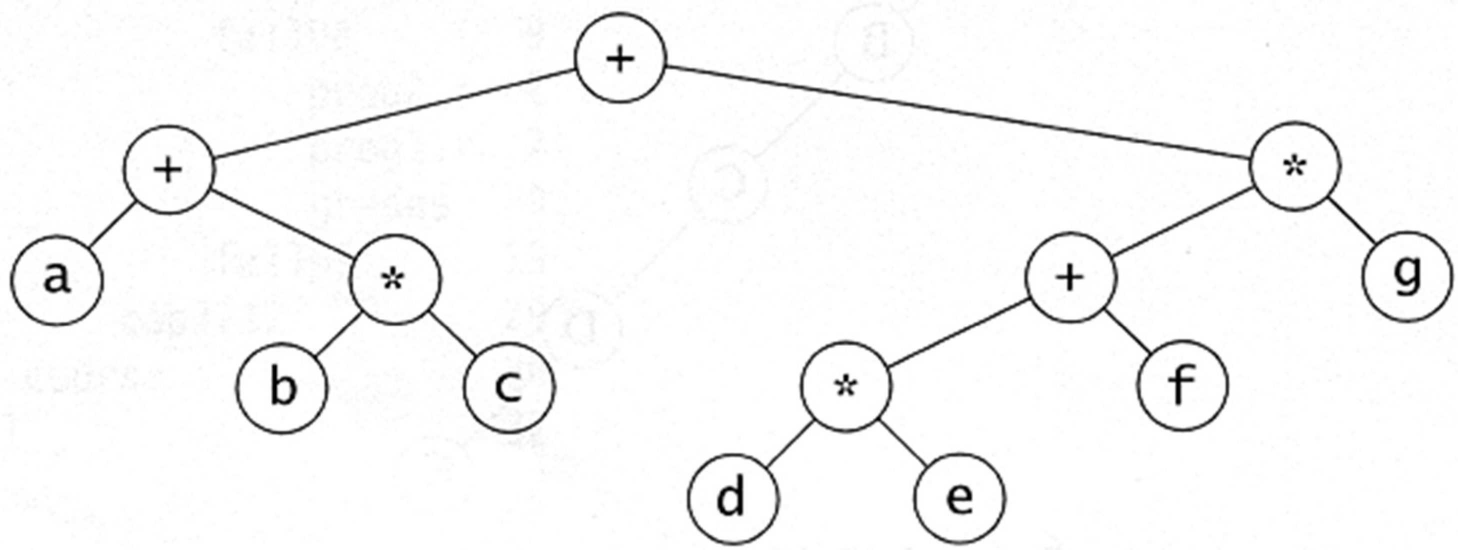


Figure 4.14 Expression tree for (a + b * c) + ((d * e + f) * g)

Preorder, Postorder and Inorder

Postorder traversal

- left, right, node
- postfix expression
  - ✓ abc*+de*f+g*+

Inorder traversal

- left, node, right.
- infix expression
  - ✓ a+b*c+d*e+f*g

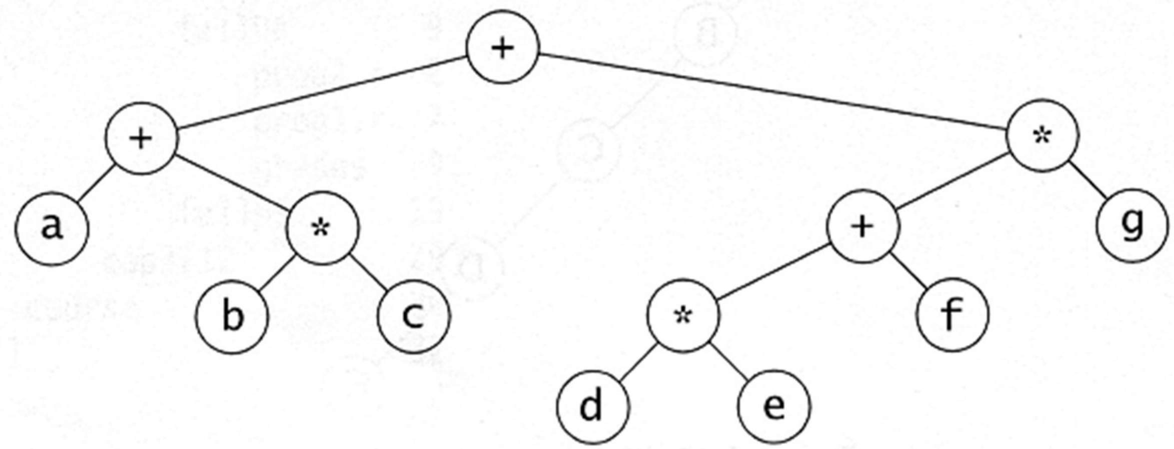


Figure 4.14 Expression tree for (a + b * c) + ((d * e + f) * g)